Combinatorial Solution to the Problem of Optimal Routing in Progressive Gradings

By V. E. BENEŠ

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The grading or graded multiple proposed by E. A. Gray is a certain kind of one-stage, two-sided, partial access telephone connecting network for switching customers' lines to trunks all having the same destination. Its essential feature is that traffic from lines not having identical access patterns can be offered to a common trunk, and so pooled. In a progressive grading the trunk groups are partially ordered in a hierarchy, i.e., some provide primary routes, others function as secondary routes which handle traffic overflowing from primary routes, as well as originating traffic, etc., up to final routes.

A call which is using an overflow or "later" trunk when it could be using a primary or "earlier" group is said to make a "hole in the multiple". It was recognized early in the development of gradings that such holes were undesirable.

The problem of optimal routing in telephone networks, considered in general in the author's earlier work, is here specialized to progressive gradings. It had been shown that for networks with certain combinatorial properties the optimal choices of routes for accepted calls (so as to minimize the loss under perfect information) could be described in a simple and intuitive way in terms of these properties. The present paper gives a proof that all progressive gradings have such a combinatorial property, associated with the hierarchical nature of the grading. The optimal policy for routing accepted calls is related to the phenomenon of "holes in the multiple", and can be paraphrased in the traditional telephone terminology thus: filling a hole in the multiple is preferable to using a final route, and filling an earlier hole is preferable to filling a later one.

I. INTRODUCTION

The term 'hierarchical' has often been used to describe connecting networks in which the possible routes for a call are ordered, with the

order determining the routing decisions in that the earlier routes are hunted over before the later. The Bell System's toll network is often cited as an example of a hierarchical network. Recently, J. H. Weber has used the word 'hierarchical' in a more technical sense to describe trunking networks ". . . in which at least some of the trunk groups are *high usage;* i.e., traffic which is not carried can be overflowed to other groups, at least some of which are *finals,* which have no alternate route."[1]

In this paper, we consider some ways in which the concept of a hierarchy of routes is relevant to the problem of optimal routing as formulated in previous work.[2] Naturally, such a hierarchy can be relevant to routing only if it is in a suitable way related to those combinatorial properties of the network which distinguish the 'good' from the 'poor' ways of completing calls. (Examples of such properties were given in Ref. 2.) It sball be shown that natural hierarchies associated with certain *gradings* hold the key to the routing problem in these one-stage networks.

It is now known[2] that if a network possesses one of certain combinatorial properties, then tbis property can be used to describe in a simple way the optimal choices of routes for accepted calls so as to minimize the loss under perfect information. The next natural question is, then, what networks possess some of these properties? We shall prove that the members of an important subclass of connecting networks, that of *progressive gradings,* all have a combinatorial property similar to the strongest of those of Ref. 2; this property is associated with a natural hierarchy of routes, and leads to a solution of the routing problem for accepted calls.

II. GRADINGS

We first discuss and clarify some of the usage and terminology associated with gradings. Since about* 1905 the noun 'grading' and the adjective 'graded' have been used in telephony to describe a certain kind of one-stage two-sided network for connecting customers' lines to trunks all having the same destination. Roughly speaking, a grading has this property: some trunk is such that two lines have access to it which do not have access to the same trunks. The essential feature is that traffic from distinguishable lines (i.e., ones not having identical access patterns) can be offered to a common trunk.

* E. A. Gray proposed the "graded multiple" in 1905, and was granted a patent for it (No. 1002388) in 1911.

It appears, though, that the word 'grading' has been used in a wider sense in Europe than in the United States. In particular, the American usage[3] implies a certain order in the pattern of access that the lines have to the trunks, whereas in the European meaning this implication is absent. The order implicit in the American usage amounts to this: the trunks are partitioned into groups which are so partially ordered that no group has more than one successor in the ordering; a line that has access to one group has access to all groups that follow it in the ordering. (This ordering usually determines the order in which the lines hunt over the trunks.) Thus, e.g., a trunk group with no predecessors in the ordering can be used by exactly one group of lines, for which it is the "primary" route. In one European sense of "grading," however, a trunk group which is the first one hunted over by one line group may be the nth one ($n > 1$) hunted over by some other line group.[4] The distinction drawn here is of some importance, inasmuch as the order structure implicit in the American usage gives rise to a natural hierarchy of routes that is directly relevant to routing, whereas in the more general case this hierarchy is not necessarily present.

Recently, in an effort to establish a uniform terminology, the nomenclature committee of the International Teletraffic Congress decided[5] that the terms 'grading' and 'graded multiple' should be interchangeable, and the structures described in R. I. Wilkinson's paper[3] as graded multiples be called, more specifically, *progressive graded multiples* or *progressive gradings*, the word 'progressive' here referring to the order structure we have described as characteristic of the American usage. The usage recommended by this committee is adopted herein.

Since the present work can be viewed as a continuation of Ref. 2, we take the liberty of assuming familiarity with the notations and concepts used there, and we include only occasional reminders of the meanings of important notions.

III. HIERARCHIES OF ROUTES

It will be convenient to have a notation for *routes*. A route r for a call c is just a way in which c can be put up or realized in a network ν, and so it can be identified with the state in which the only call in progress is c using route r. Thus, a route for c is any element of $\gamma^{-1}(c)$.* We use the variables q and r (over the set L_1 of states with one call in progress) to denote routes.

* We recall that if x is a state, $\gamma(x)$ is the assignment of inlets to outlets realized by x.

By a *hierarchy of routes* we mean a partial ordering $\supseteq$ contained in

$$\bigcup_{c} [\gamma^{-1}(c)]^2.$$

It is apparent that $\supseteq$ can hold only between alternative routes for the same call. (Of course, not every hierarchy of routes is relevant to routing; only those that have a suitable relation to the ways in which calls in progress block new calls will be of interest. The problem is to clarify the meaning of 'suitable'.)

A hierarchy of routes, being a partial ordering of the states with one call in progress, can be extended to, or can induce, a partial ordering of the whole set S of states in several natural ways. Since $\supseteq$ can hold only between alternative routes for the same call, it is reasonable to confine attention to extensions which hold only between states that are equivalent in the sense of $\sim$ in Ref. 2, i.e., are (possibly) different ways of realizing the same assignment. An obvious first candidate for such an extension is given by the condition

$$x \sim y \quad \text{and} \quad r \leqq x,\, q \leqq y,\, r \sim q \quad \text{imply} \quad r \supseteq q. \tag{1}$$

However, we eschew this definition in favor of a stronger one: let us set

$x \supseteq y \equiv x$ is reachable from y by sequentially moving calls in progress from routes that are lower (later) (in the sense of $\supseteq$ on L_1) to routes that are higher (earlier).*

It is intended here not merely that, as in (1), each call have a higher route in x than in y, but that it should be possible to pass from y to x by a sequence of equivalent states each differing from the previous one in that one call has been rerouted on a higher route. This stronger condition is rendered formally by first defining

$$\begin{aligned} x\,Q\,y \equiv |\, x \cap y \,| = |\, x \,| - 1 &\text{ and either} \\ x - (x \cap y) \supseteq y - (x \cup y) &\text{ or} \\ |\, x \,| = 1 \text{ and } x \supseteq y & \end{aligned}$$

and then setting

$$\begin{aligned} \supseteq &= I \cup Q \cup Q^2 \cup \cdots \\ &= \text{transitive closure of } Q. \end{aligned} \tag{2}$$

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* In an attitude prejudiced and justified by the principal results (Theorems 1 and 2) we are working toward, we use the words 'lower', 'earlier', and their antonyms so as to suggest consistently that *lower* routes are less desirable than higher, *earlier* ones are preferable to later, etc.

IV. PROGRESSIVE GRADINGS

In a one-stage connecting network $\nu = (G,I,\Omega,S)$, with I the set of customers' lines (inlets) and Ω that of trunks (outlets), the graph G giving network structure is determined entirely by the access relation A such that

$$lAt \equiv \text{line } l \text{ has access to trunk } t.$$

The set S of states of ν can be represented by the set of all subsets of A which are one-to-one correspondences. The range of x, rng (x), is the set of trunks which are busy in x.

The access relation A can be used to give a simple definition of a progressive grading. We use $X \times Y$ for the Cartesian product of X and Y, i.e., the set of pairs (x,y) with $x \varepsilon X$ and $y \varepsilon Y$. If X is a set, $| X |$ denotes the number of elements of X.

Definition: ν is a *progressive grading* if and only if it is a one-stage network for which there exist partitions Π and Ξ of Ω and I, respectively, and a partial ordering $\geqq$ of Π, such that for T, U, $V \varepsilon \Pi$ and $L \varepsilon \Xi$

(*i*) $(L \times T) \cap A \neq \theta$ implies $(L \times T) \subseteq A$,

(*ii*) $(L \times U) \subseteq A$, $V \geqq U$ imply $(L \times V) \subseteq A$,

(*iii*) $U \geqq T$, $V \geqq T$ imply $U \leqq V$ or $V \leqq U$

(*iv*) $| L | \geqq | \bigcup_{T:(L \times T) \subseteq A} T |$.

The first condition simply says that if a line has access to some trunk from a group T, then all lines in its line group have access to every trunk in T. The second condition says (roughly) that a line with access to a trunk group T has access to all groups that are later than T in the partial ordering. The third condition says that a trunk group is followed (in the partial ordering) by at most one other group; if the "later" groups are thought of as overflow groups, this means that each group has at most one group to which to overflow traffic. Finally, the fourth condition rules out the relatively uninteresting cases in which some line group has access to more trunks *in toto* than there are lines in the group.

It is apparent that if a trunk group T_1 is later than one T_2, then every line with access to T_2 has access to T_1. This is the "progressive" property. In analogy with the intuitions expressed in Ref. 2, it should be better to use an earlier trunk group than a later one, if both are available. Thus, the structure of a progressive grading at once suggests the conjecture that optimal routing will consist of using the early routes in

preference to the later or (to anticipate a bit) overflow groups. This conjecture is true and follows from Theorem 2. In traditional telephone terminology (see E. C. Molina's appendix in Ref. 3) it states that filling a hole in the multiple is preferable to using a final route, and that filling an earlier hole is preferable to filling a later one.

A line group L is said to be a *bye* if it has access only to "overflow" trunk groups, i.e., if

$$\inf_{\leqq} \{T{:}LAT\}$$

is not minimal in $\leqq$, where we have written LAT for $(L \times T) \subseteq A$.

It is easily seen that in a progressive grading a hierarchy of routes can be defined by this rule: $r \supseteq q$ if and only if $r \sim q$ and $g(q) \geqq g(r)$, where $g(r)$ is the trunk *group* used by route r.* This is the *natural hierarchy* of routes associated with a progressive grading; here $r \supset q$ if and only if $r \sim q$ and r is on an "earlier" trunk group than q. In this instance, $\geqq$ is also a *simple* ordering on each $g(\gamma^{-1}(\gamma(r)))$. These simple orderings forming the hierarchy of course correspond exactly to the preference relation among routes suggested by the natural intuition (already mentioned) that there is no point in using a later or "overflow" trunk when an earlier one is available, because possibly fewer lines have access to the latter. The relation $\supseteq$ defined above on L_1 extends by (2) to all of S.

V. PARTIAL ORDERING OF PROGRESSIVE GRADINGS

In a proof to be given later we shall use the fact that the set of progressive gradings can be partially ordered by a relation $\supseteq$ according to the following definition of covering: ν_1 covers ν_2 if and only if ν_2 is obtained from ν_1 by removing, for some line group L, either (case 1) a trunk from the first (in $\leqq_1$) trunk group to which L has access together with one line of L if L has access to more than one trunk, or (case 2) *the* trunk to which L has access together with L itself if L has access to exactly one trunk. That is, if ν_1 is defined by partitions Π_1, Ξ_1, a partial ordering $\geqq_1$ of Π_1, and an access relation A_1, then ν_1 covers ν_2 provided that there exist $t \in \Pi_1$ and $l \in L \in \Xi_2$ with

$$T = \inf_{\geqq_1} \{U \in \Pi_1{:}(L \times U) \subseteq A\}$$

such that ν_2 is defined by (case 1)

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* Note the shift to the converse.

$$\Pi_2 = \Pi_1 - \{T\} + \{T - \{t\}\}$$
$$\Xi_2 = \Xi_1 - \{L\} + \{L - \{l\}\}$$
$$\geqq_2 = \geqq_1 \quad \text{with} \quad T - \{t\} \quad \text{for } T \text{ throughout}$$
$$A_2 = A_1 - (I \times \{t\}) - (\{l\} \times \Omega),$$

if $T \neq \{t\}$ or $A_1 \cap (L \times \Omega) \not\subseteq (L \times \{t\})$, and by (case 2)

$$\Pi_2 = \Pi_1 - \{T\}$$
$$\Xi_2 = \Xi_1 - \{L\}$$
$$\geqq_2 = \geqq_1 - (\Pi_1 \times \{T\})$$
$$A_2 = A_1 - (I \times \{t\}) - (L \times \Omega),$$

if $T = \{t\}$ and $A_1 \cap (L \times \Omega) \subseteq (L \times \{t\})$.

For practical purposes a network in which some line group has access to no trunks is in all respects equivalent to the same network with those lines omitted. For this reason the definition of covering was divided into cases 1 and 2, so as to huild this equivalence right into the definition.

As we have said, ν_1 covers ν_2 if and only if ν_2 results from ν_1 hy ripping out (*i*) some trunk from a "primary" group, (*ii*) a line with access to it, and (*iii*) all crosspoints associated with these terminals, with the proviso that if this leaves some lines with access to no trunks, then these lines are also to he removed. Because of this, there exists a natural or canonical map μ of the states $S(\nu_1)$ of ν_1 into those $S(\nu_2)$ of ν_2, defined roughly by the condition that μx is what is left of x after the line and trunk that define the covering of ν_2 by ν_1 bave heen ripped out. The canonical map can be defined formally very simply, as follows: A state x of ν_1 is representable as a suhset of A_1 which is also a one-to-one correspondence; similarly, a state of ν_2 is just a one-to-one map contained in A_2; what is left of x after the ripping-out process is just

$$\mu x = x \cap A_2 .$$

Thus, if μ corresponds to ripping out line l and trunk t, and $x = \{(l,t)\}$, then $\mu x = \theta =$ zero state. If $x = \{(l,t_1)\}$ or $x = \{(l_1, t)\}$ with $l_1 \neq l$ and $t_1 \neq t$, then again $\mu x =$ zero state. If $x = \{(l_1, t_1)\} \cup y$, with $l_1 = l$ or $t_1 = t$, then $\mu x = \mu y$. It is easy to see that if μ rips out l and t, then μS is isomorphic with the "cone"

$$\{x \varepsilon S{:}x \geqq \{(l,t)\}\},$$

because it does not make any difference whether l and t are present in the system and connected to each other, or are just ahsent. That is,

μS is essentially the set of states of S that remain available if l is connected to t with a holding-time of $+\infty$.

This notion of a canonical map provides many useful notations. It is convenient to extend the μ-notation as follows: For $T \varepsilon \Pi$

$$\mu T = T \cap \text{range}\,(A_2) = \begin{cases} T - \{t\} & \text{if} \quad t \varepsilon T \quad \text{and} \quad T \neq \{t\}, \\ T & \text{if} \quad t \not\varepsilon T, \\ \theta & \text{if} \quad T = \{t\}. \end{cases}$$

Clearly, μT is what is left of the trunk group T after the line l and the trunk t associated with μ have been ripped out. Also, we set

$$\mu \geqq \; = \{(\mu T_1, \mu T_2): T_1 \geqq T_2, \mu T_1 \neq \theta, \text{ and } \mu T_2 \neq \theta\}$$

$$\mu \supseteq \; = \{(\mu x, \mu y): x \supseteq y, \; \mu x \neq \theta \text{ or } x = \theta, \text{ and } \mu y \neq \theta \text{ or } y = \theta\}.$$

The relation $\mu \geqq$ can be seen to be identical with $\geqq_2$; it is a useful mnemonic; it defines the hierarchy of routes in the "reduced" system ν_2; the partial ordering induced in $S(\nu_2)[= \mu S(\nu_1)]$ by this hierarchy is precisely $\mu \supseteq$.

VI. PRELIMINARY RESULTS

In Ref. 2, for a general partial ordering R, the notation

$$\sup_R A_{cx}$$

was used for the *set*

$$\{y: z \; \varepsilon \; A_{cx} \text{ implies } yRz\} \cap A_{cx}$$

whenever this set was nonempty. The notation was chosen to denote a *set* of R-maximal elements of A_{cx}, rather than an actual R-maximal element itself, so as not to prejudge the question as to how many there were. It will be shown that if the network ν under study is a progressive grading, and $R = \supseteq =$ natural hierarchy, then unless c is blocked in x (and A_{cx} is empty) A_{cx} always has a $\subseteq$-maximal element which is unique to within equivalence under permutations of lines within their line groups and trunks within their trunk groups.

Let now x be a state and let $c \; \varepsilon \; x$ be a call which is not blocked in x. It is apparent that for $y, z \; \varepsilon \; A_{cx}$ we have either $g(y - x) \geqq g(z - x)$ or $g(y - x) \leqq g(z - x)$. Hence, there is a $y_0 \; \varepsilon \; A_{cx}$ such that

$$g(y_0 - x) \geqq g(w - x)$$

$$y_0 - x \supseteq w - x$$

$$y_0 \, Q \, w$$

$$y_0 \supseteq w$$

for all $w \varepsilon A_{cx}$, and y_0 is unique to within equivalence. (Recall the construction of Q in Section III, and the fact that $\supseteq$ is $\overline{I \cup Q}$.) Hence,

$$\sup_{\supseteq} A_{cx} \quad (\sup A_{cx} \text{ for short when the context permits})$$

exists, and equals $\tau(y_0)$, $\tau(\cdot)$ being the natural homomorphism of S into the quotient $S/(\supseteq \cap \subseteq)$. (See Ref. 2.)

We now consider policies $\varphi(\cdot,\cdot)$ such that

$$\varphi(e,x)\begin{cases} = x - h & \text{if } \quad e \text{ is a hangup } h, \\ \varepsilon \sup A_{cx} & \text{if } \quad e \text{ is a new call } c \text{ not blocked in } x. \end{cases} \tag{3}$$

Such a policy expresses the routing rule of always choosing the earliest available trunk in the natural hierarchy characteristic of a progressive grading.

The relation B (for "better") was defined in Ref. 2 by the condition

$x \; B \; y$ if and only if $x \sim y$ and every call blocked in x is also blocked in y.

By Theorem 1, to be proved shortly, it will follow that $x \supseteq y$ implies $x \; B \; y$, which in turn implies $s(x) \geqq s(y)$. Thus, the policies $\varphi(\cdot,\cdot)$ coincide with the "maximum $s(\cdot)$" policies suggested in Ref. 2. (See Ref. 2 for notations.)

Lemma 1: If the line of c is not involved in the canonical map μ, and $A_{cx} \neq \theta$, then

$$\mu(\sup_{\supseteq} A_{cx}) \subseteq \sup_{\mu\supseteq} A_{c(\mu x)} .$$

Proof: Let l^* be the line of c, and suppose that

$$y \; \varepsilon \sup_{\supseteq} A_{cx} .$$

Let l and t be the line and trunk, respectively, associated with μ. There exists a trunk t^* such that

$$y = x \cup \{(l^*,t^*)\}$$

$$\mu y = \mu x \cup \{(l^*,t^*)\}$$

$$t^* \; \varepsilon \inf_{\geqq} \{T{:}T \not\subseteq \operatorname{rng}(x) \text{ and } l^*AT\}.$$

Let T^* denote the set (trunk group) achieving the infimum on the right. Since t is busy in x and t^* is not, $t \neq t^*$. Thus, $T^* \neq \{t\}$, and $\mu(T^*) \neq \theta$.

We first observe that l^*AT implies $l^*A_2\mu T$, since c is not involved in μ.

We next show that $\mu T \nsubseteq \text{rng}\,(\mu x)$ implies $T \nsubseteq \text{rng}\,(x)$. If not, then there exists $t_1 \,\varepsilon\, \mu T$, hence $\varepsilon\, T$ sucb that $t_1 \not\varepsilon\, \text{rng}\,(\mu x)$ and $t_1 \,\varepsilon\, \text{rng}\,(x)$. But

$$\text{rng}\,(\mu x) = \text{rng}\,(x) - \{t\}.$$

Hence, $t_1 = t =$ trunk removed by μ. But this is impossible since $t_1 \,\varepsilon\, \mu T$, while $t \not\varepsilon\, \mu T$.

Now $T^* \leqq T$ for every T such that $T \nsubseteq \text{rng}\,(x)$ and l^*AT. From the two previous paragraphs, it follows that

$$(\mu T^*)(\mu\leqq)\mu T$$

for every T such that $\mu T \nsubseteq \text{rng}\,(\mu x)$, $l^*A\mu T$, $\mu T \neq \theta$. That is,

$$\mu T^* = \inf_{\mu\leqq} \{T{:}T \nsubseteq \text{rng}\,(\mu x),\ l^*AT\}.$$

Now $t^* \,\varepsilon\, T^*$, $t^* \neq t$, so $t^* \,\varepsilon\, \mu T^*$. If now $w \,\varepsilon\, A_{c(\mu x)}$, tben

$$(w - \mu x)(\mu\subseteq)\{(l^*,t^*)\}$$

$$w(\mu\subseteq)(\mu x \cup \{(l^*,t^*)\})$$

$$w(\mu\subseteq)\mu y.$$

Thus,

$$\mu y \,\varepsilon \sup_{\mu\subseteq} A_{c(\mu x)}\,,$$

and since y was arbitrary within $\sup\limits_{\supseteq} A_{cx}$, the lemma is proved.

Lemma 2: In a progressive grading, $Q \subseteq B$.

Proof: Let $x\,Q\,y$. This implies that there exists $z \,\varepsilon\, B_x \cap B_y$ such that $x - z \supseteq y - z$, i.e.,

$$g(y - z) \geqq g(x - z).$$

Now let c be a call from line l which is blocked in x but not in y. Tben c is not blocked in z either. Tbe only trunk which is busy in x and not in z is that used by the call $\gamma(x - z)$. Tbus, since c is blocked in x and not in z, $g(x - z)$ is a trunk group usable for the call c. However, by property (*ii*) of progressive gradings, $\{l\} \times g(y - z) \subseteq A$, i.e., l bas access to the group $g(y - z)$ as well. Hence, some trunk of $g(y - z)$ is idle in x, since the call $\gamma(x - z)$ has a choice of routes in state z, one of these being on $g(y - z)$. Thus, c is not blocked in x, and $x\,B\,y$.

Theorem 1: In a progressive grading, the partial ordering $\supseteq$ induced by the natural hierarchy of routes is contained in B.

Proof: Immediate from Lemma 2 and the facts that $\supseteq$ is the transitive closure of $I \cup Q$, and that B is transitive.

Lemma 3: If $x \supseteq y$, then x is obtainable from y by moving calls to earlier routes in such a way that each call is moved at most once.

Proof: The result is true if only one move is made. Suppose it to hold if n moves *in toto* are made. Let x be obtainable from y by sequence of $(n + 1)$ moves. The trunk groups available for a given call c form a set simply ordered by $\leqq$, and so can be indexed $1, 2, \cdots$, the $\leqq$-earlier receiving the lower integer. For $c \leqq \gamma(x)$, let $n(c,x)$ be the index of the group used by c in x. Some call c that is moved in obtaining x from y achieves

$$\min \{n(c_1, x) \mid c_1 \text{ moved in getting } x \text{ from } y\}.$$

Starting in state y it is possible to move such a call (once) directly to its route in x, to get a state z in which it is still possible to carry out exactly each of the moves that take y into x except those involving c. These are at most n in number, so each call involved need be moved at most once.

A policy $\varphi(\cdot,\cdot)$ is said to *preserve* a relation $R \subseteq \sim$ if $x \; R \; y$ implies

$$\varphi(e,x) \; R \; \varphi(e,y)$$

for every event e that is either a hangup or a new call not blocked in either x or y. It has been shown in Ref. 3 for a general network that if φ preserves B then it embodies the optimal routing policy for accepted calls.

The main theorem we prove (Section VII) states that a sup A_{ex} policy, i.e., one satisfying (3), preserves $\supseteq$. The method to be used in the proof of this result is illustrated in part by the following remarks: consider linear arrays x, y, z, $\cdots$ each of n urns, $n \geqq 2$, each urn containing at most one ball, with fewer than n nonempty urns per array. Let $x \supseteq y$ mean that x is obtainable from y by moving balls to the left. Let φx denote the result of adding a ball in the leftmost empty urn.

Observation: If $x \supseteq y$, then $\varphi x \supseteq \varphi y$.

Proof: The result is obviously true for $n = 2$ by enumeration. Let it hold for a given value $n \geqq 2$, and consider arrays x, y of n urns satisfying the hypotheses. Let ψz denote the result of removing the leftmost urn from z, and bz that of adding an urn containing one ball at the left of z. There are two cases: (i) the leftmost urns are empty in both x and

y, or both nonempty in x, y; (*ii*) in y, but not in x, the leftmost urn is empty.

Case (*i*): $\varphi x = b\psi x$, $\varphi y = b\psi y$, $\psi x \supseteq \psi y$; hence, $\varphi x \supseteq \varphi y$.

Case (*ii*): In obtaining x from y some ball moved into the leftmost urn. Obtain z from y by moving just this one ball to the leftmost urn. Then $x \supseteq z \supseteq y$, $\varphi x \supseteq \varphi z$, $\varphi y = b\psi z$, $\varphi z = b\varphi\psi z$. Since $\varphi\psi z$ is obtained from ψy by removing some ball, and replacing it in the leftmost empty urn of the resulting array, we have $\varphi\psi z \supseteq \psi y$, and so $\varphi z \supseteq \varphi y$.

In cases 3 and 4 of the proof of the next theorem, the analog of the inductive index n will be the partial ordering of the set of progressive gradings.

VII. PRINCIPAL RESULT

Theorem 2: In a progressive grading ν let $\supseteq$ be the partial ordering induced by the natural hierarchy of routes in ν, and let φ be a policy with the property that

$$\varphi(c,x) \in \sup_{\supseteq} A_{cx}, \qquad c \in x,\ c \text{ not blocked in } x.$$

Then φ preserves $\supseteq$.

Proof: The proof is by induction over the partial ordering $\supseteq$ of the set of progressive gradings which is defined by the definition of covering given earlier. A grading ν that is minimal in $\supseteq$ has no "overflow groups", i.e., $\geqq$ = identity relation, so that no trunk group has a successor in the order $\geqq$ characteristic of ν. Thus, ν consists entirely of trunk groups serving line groups on a one-to-one basis, so that for some n

$$A = \bigcup_{i=1}^{n} (L_i \times T_i),$$

where

$$\Xi = \{L_i\ , = 1, \cdots, n\}$$

$$\Pi = \{T_j\ , = 1, \cdots, n\}, \quad \text{with} \quad |\,T_j\,| = 1.$$

In this minimal case $\supseteq$ is the identity relation, and φ obviously preserves it.

As a hypotheses of induction, we now suppose that every progressive grading covered by ν has the property that any sup A_{cx}-policy preserves $\supseteq$. Let now $x \supseteq y$ in ν and let $c \in x$. The induction argument will have four cases, the last two of which are analogous to the observation made earlier.

Case 1: $x \supseteq y$, and e is a hangup h. There is a sequence $x = z_1, z_2, \cdots, z_n = y$ with

$$z_j \, Q \, z_{j+1} \qquad j = 1, \cdots, n-1.$$

This sequence indicates how one would get y from x by moving calls to "preferred" routes. By Lemma 3 it is no restriction to assume that no call is rerouted more than once. Let the route of h be r in x and q in y. If h is one of the calls whose route is changed in the above sequence, say to take z_k into z_{k+1} by changing the route of h from r to q, then

$$x - r = z_1 - r, z_2 - r, \cdots, z_k - r = z_{k+1} - q, \cdots, z_n - q = y - q$$

is a sequence which shows that $(x - r) \supseteq (y - q)$. If the route of h is not changed, then $r = q$ and the same conclusion follows.

Case 2: $x \supseteq y$, and $e \, \varepsilon \, x$ is a new call c blocked in x. By Theorem 1, $x \, B \, y$, so c is also blocked in y. Then,

$$A_{cx} = \{x\}, \qquad A_{cy} = \{y\}$$

$$\varphi(c,x) = x \qquad \varphi(c,y) = y$$

$$\varphi(c,x) \supseteq \varphi(c,y).$$

Case 3: $x \supseteq y$, e is a new call c not blocked in either x or y, and the line group L of c is not a bye. Let

$$T = \inf_{\leq} \{S{:}LAS\}.$$

Subcase 3.1: T is full in neither x nor y. Then there exist routes r, q such that $g(r) = g(q) = T$,

$$\varphi(c,x) = x \cup r, \qquad \varphi(c,y) = y \cup q,$$

$r \equiv q$ modulo trunk permutations within T, and clearly

$$\varphi(c,x) \supseteq \varphi(c,y)$$

since c was put up on group T in both cases. To see this, if $x = z_1, z_2, \cdots, z_n = y$ is a sequence with

$$z_j \, Q \, z_{j+1} \qquad j = 1, \cdots, n-1,$$

showing that $x \supseteq y$, then
$x \cup r = z_1 \cup r, z_2 \cup r, \cdots, z_n \cup r = y \cup r, y \cup q$ is a sequence which shows that

$$(x \cup r) \, Q \, (y \cup q).$$

This is because we can assume without loss of generality that the transformations which change y into x reroute a call at most once, and thus move no calls onto T. (Lemma 3.)

Subcase 3.2: T is full in both x and y. Since L is not a bye, there exist l, m ε L and t, u ε T with

$$(l,t) \ \varepsilon \ x \quad \text{and} \quad (m,u) \ \varepsilon \ y.$$

Because l, m and t, u are respectively interchangeable, i.e., since lines and trunks are permutable within their respective groups, no loss of generality is incurred if it is supposed that $l = m$ and $t = n$. Let μ be the canonical map corresponding to ripping out l and t.

Then ν covers ν_1, where ν_1 is defined by ripping l and t out of ν, i.e., by

$$\mu\Pi = \Pi_1 = \begin{cases} \Pi - \{T\} + \{T - \{t\}\} & \text{in case 1} \\ \Pi - \{T\} & \text{in case 2,} \end{cases}$$

$$\mu\Xi = \Xi_1 = \begin{cases} \Xi - \{L\} + \{L - \{l\}\} & \text{in case 1} \\ \Xi - \{L\} & \text{in case 2,} \end{cases}$$

$$\mu\geqq = \geqq_1 = \begin{cases} \geqq \quad \text{with } T - \{t\} \text{ replacing } T \text{ throughout,} & \text{in case 1} \\ \geqq - (\Pi_1 \times \{T\}), & \text{in case 2,} \end{cases}$$

$$\mu A = A_1 = \begin{cases} A - (I \times \{t\}) - (\{l\} \times \Omega) & \text{in case 1} \\ A - (I \times \{t\}) - (L \times \Omega) & \text{in case 2,} \end{cases}$$

with

$$\text{case } 1 \equiv T \neq \{t\} \quad \text{or} \quad A \cap (LT) \nsubseteq (L \times \{t\})$$

$$\text{case } 2 \equiv T = \{t\} \quad \text{and} \quad A \cap (LT) \subseteq (L \times \{t\}).$$

The line of c is not involved in μ, and $A_{cx} \neq \theta$, $A_{cy} \neq \theta$. Hence, Lemma 1 gives

$$\mu\varphi(c,x) \ \varepsilon \sup_{\mu\supseteq} A_{c(\mu x)}$$

$$\mu\varphi(c,y) \ \varepsilon \sup_{\mu\supseteq} A_{c(\mu y)} \ .$$

Since $x \supseteq y$, and either both $\mu x = 0$, $\mu y = 0$, or neither, we have

$$(\mu x,\mu y) \ \varepsilon \ \mu\supseteq. \tag{4}$$

Let ξ be a policy for ν_1 with

$$\xi(d,\mu z) \ \varepsilon \sup_{\mu\supseteq} A_{d(\mu z)} \ , \qquad Vd \ \varepsilon \ \mu z. \tag{5}$$

The hypothesis of induction and (4) give

$$\xi(c,\mu x)(\mu\supseteq)\xi(c,\mu y). \tag{6}$$

However, by (6) and (5)

$$\begin{aligned}\mu\varphi(c,x)(\mu\supseteq)&\xi(c,\mu x)\\ (\mu\supseteq)&\xi(c,\mu y)\\ (\mu\supseteq)&\mu\varphi(c,y).\end{aligned}$$

But $\varphi(c,x)$ differs from $\mu\varphi(c,x)$ and $\varphi(c,y)$ from $\mu\varphi(c,y)$, only in having an additional line l and an additional trunk t connected to each other. Hence,

$$\varphi(c,x) \supseteq \varphi(c,y).$$

The argument of subcase 3.2, basic to Theorem 2, can be appreciated by looking at it thus: $x \supseteq y$ means $\exists z_1, \cdots, z_n$ with $z_i \; Q \; z_{i+1}$, $i = 1, \cdots, n-1$, $z_1 = x$, $z_n = y$. Since

$$e = \{(l,t)\} \leqq x \cap y$$

we have $r \leqq z_i$, $i = 1, \cdots, n$ because we can assume that the call using r is not moved as y is transformed into x by moving calls. Thus,

$$\begin{aligned}&(z_i - r) \; Q \; (z_{i+1} - r), \qquad i = 1, \cdots, n-1\\ &(x - r) \; Q \; (y - r).\end{aligned}$$

But the "cone" $\{z{:}z \geqq r\}$ is isomorphic to the states of a grading (ν_1 of the proof) covered by ν and the isomorphism, *viz.*, μ restricted to the cone, has the basic property, for x, y in the cone

$$x \supseteq y \quad \text{if and only if} \quad (\mu x)(\mu\supseteq)(\mu y).$$

Subcase 3.3: T is full in x, but not in y. Since L is not a bye it is $\leqq$-minimal, and hence there exists a call d with $d \leqq \gamma(x) \cap \gamma(y)$ such that d is on T in x is not on T in y, and can be moved to T in state y to give rise to a new state z without rendering impossible any the remaining moves which transform y into x. Thus, $x \supseteq z \supseteq y$. Since $x \cap z \neq \theta$, subcase 3.1 gives $\varphi(c,x) \supseteq \varphi(c,z)$. Further, the route of c in $\varphi(c,z)$ is no higher (later) in $\leqq$ than the one in y left by d as it was moved to T to give rise to state z. Hence, to within equivalence

$$\varphi(c,z) \supseteq \varphi(c,y).$$

Case 4: $x \supseteq y$, e is a new call c not blocked in either of x or y, and the line group of c is a bye. There is at least one other line group L which

is not a bye. Let

$$T = \inf_{\leqq} \{S{:}LAS\}.$$

Subcase 4.1: $L \times T \cap x \neq \theta$, $L \times T \cap y \neq \theta$ or $L \times T \cap x = \theta$, $L \times T \cap y = \theta$. Since L is not a bye, there exist l, $m \in L$ and t, $u \in T$ with

$$(l,t) \in x, \qquad (m,u) \in y$$

or

$$(l,t) \notin x, \qquad (m,u) \notin y.$$

(In the second instance, property (*iv*) of the definition of a progressive grading has been used to conclude that there must be idle lines on L if there are idle trunks on T.)

As in subcase 3.2, no loss of generality is incurred if it is supposed that $l = m$ and $t = u$. Let μ be the canonical map corresponding to ripping out l and t. The argument now continues as in subcase 3.2.

Subcase 4.2: $(L \times T) \cap x \neq \theta$, $(L \times T) \cap y = \theta$. Since L is $\leqq$-minimal, there exists a call d with $d \leqq \gamma(x) \cap \gamma(y)$ such that d is on T in x, is not on T in y, and can be moved to T in state y to give rise to a new state z without rendering impossible any of the remaining moves which transform y into x. Thus, $x \supseteq z \supseteq y$. Since $x \cap z \neq \theta$, subcase 3.1 gives $\varphi(c,x) \supseteq \varphi(c,z)$.

Let l^* be the line of c, r be the route of d in y, $T_d = g(r)$, and

$$T_c = g(\inf_{\leqq} \{S{:}l^*AS, S \nsubseteq \operatorname{rng}(y)\}).$$

Here T_c is the earliest group c could be put on in y. Let also ψ denote the operation of moving d from T_d to T, and for any call f

$$\begin{aligned} A_f &= \{g(r){:}f = \gamma(r)\} \\ &= \{S{:}\text{the line of } f \text{ has access to } S\}. \end{aligned}$$

Case (i): $T_d \in A_c \cap A_d$, $T_d \leqq T_c$. Then moving d from T_d to T means that c can use T_d in z, so $\varphi(c,z) \supseteq \varphi(c,y)$, because $\varphi(c,z)$ results from $\varphi(c,y)$ by moving first d to T_d and then c to T_d, so actually

$$\varphi(c,z) \supseteq \psi\varphi(c,y).$$

Case (ii): $A_c \cap A_d = \theta$, or $A_c \cap A_d \neq \theta$ and either

$$T_c, T_d \in A_c \,\Delta\, A_d,$$

or

$$T_d \; \varepsilon \; A_c \cap A_d , \qquad T_c \; \varepsilon \; A_c \, \Delta \, A_d ,$$

or

$$T_c \; \varepsilon \; A_c \cap A_d , \qquad T_d \; \varepsilon \; A_c \, \Delta \, A_d ,$$

or

$$T_c , T_d \; \varepsilon \; A_c \cap A_d , \qquad T_c < T_d .$$

In all these cases $\psi\varphi(c,y) = \psi(c,\psi y) = \varphi(c,z)$, whence

$$\varphi(c,z) \supseteq \varphi(c,y).$$

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